Helical Attractors in Contact Manifolds

A Numerical and Formal Study of the dm3 System

4/19/2026

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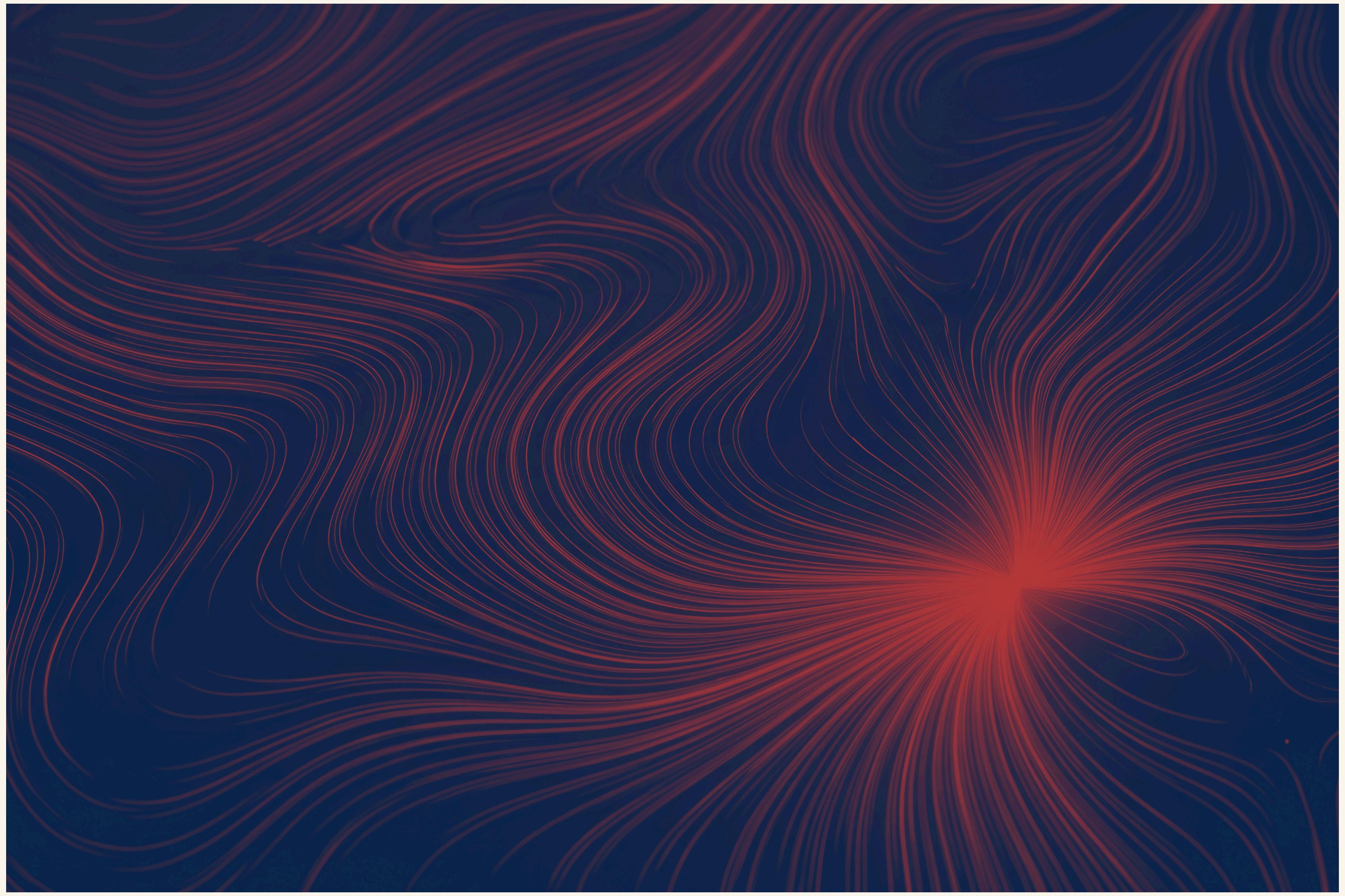
Agenda

1. Introduction and System Overview
2. Dynamics and Numerical Analysis
3. Formal Verification and Future Directions

Introduction and System Overview



What Are Contact Manifolds?

Defining Contact Manifolds

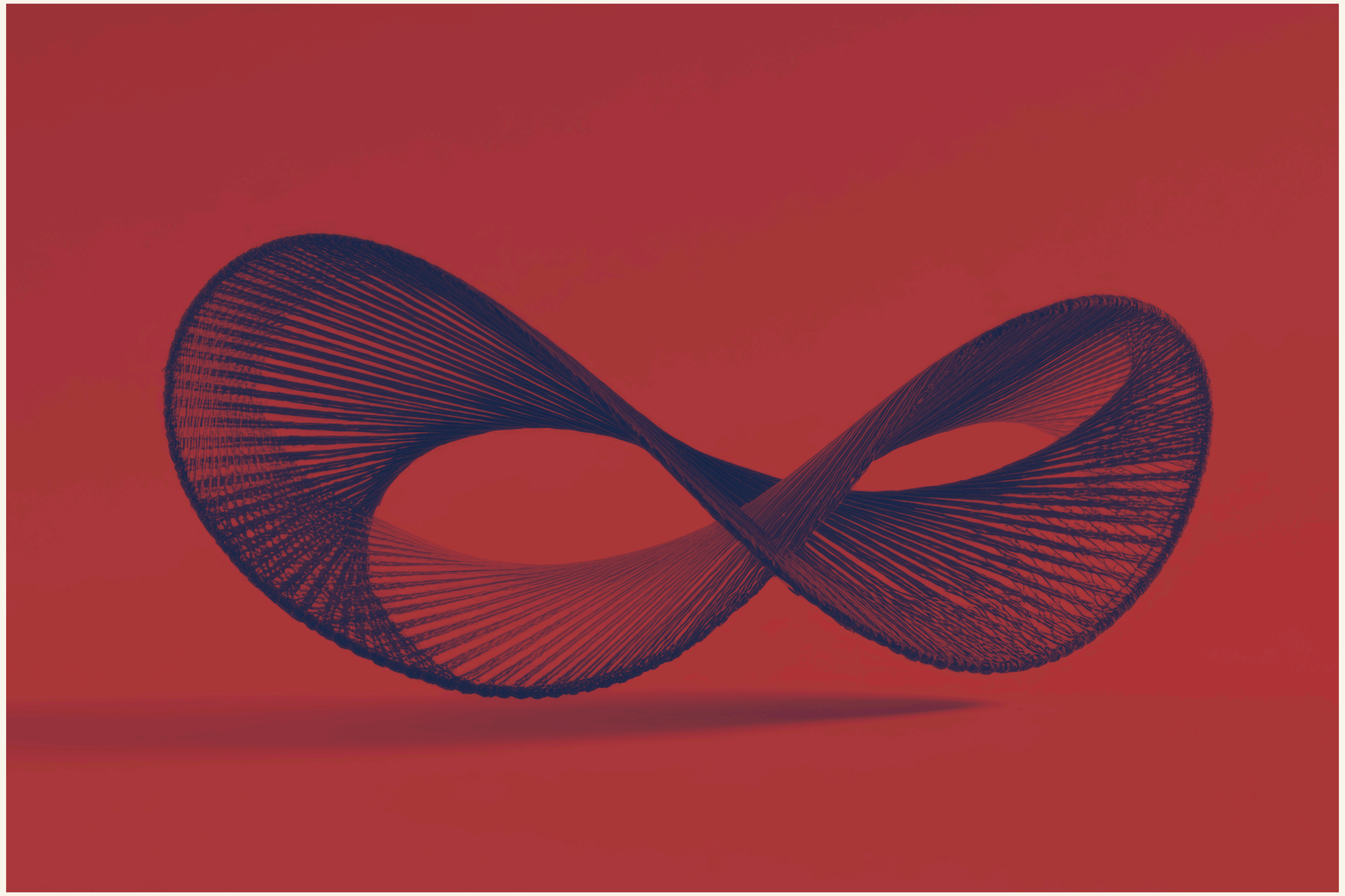
- Three-dimensional spaces with a **non-integrable plane field** at each point
- Mathematically described as (M, ξ), where $\xi = \ker \alpha$ and $\alpha \wedge d\alpha \neq 0$

Cylindrical Coordinates Example

- Prototype: $\mathbf{R^3}$ with coordinates (r, θ, z)
- Contact form: $\alpha = dz - r^2\, d\theta$

Reeb Vector Field

- Special vector field associated with the contact form
- In this setting: $\mathbf{R = \partial z}$ (vertical direction)
- Drives the dynamics and shapes limit sets like helices



Meet the dm3 System

System Overview

- Three-dimensional ODE defined on a **contact manifold**
- Uses **cylindrical coordinates**: r (radius), θ (angle), z (height)

Mathematical Structure

- Radial equation follows the **Hopf normal form**: $r(1 - r^2)$
- Includes an **exponential coupling**: ε(r – 1)e^{-z}

Key Variables

- r: radial distance from the axis
- θ: angular coordinate (rotation)
- z: height (vertical direction)

Dynamics and Numerical Analysis

The Helical Attractor Revealed

01

Helical Limit Set on Contact Manifold

- The system's limit set is a **helix**: the Reeb orbit on the unit cylinder $r = 1$.

02

Spiraling Trajectories

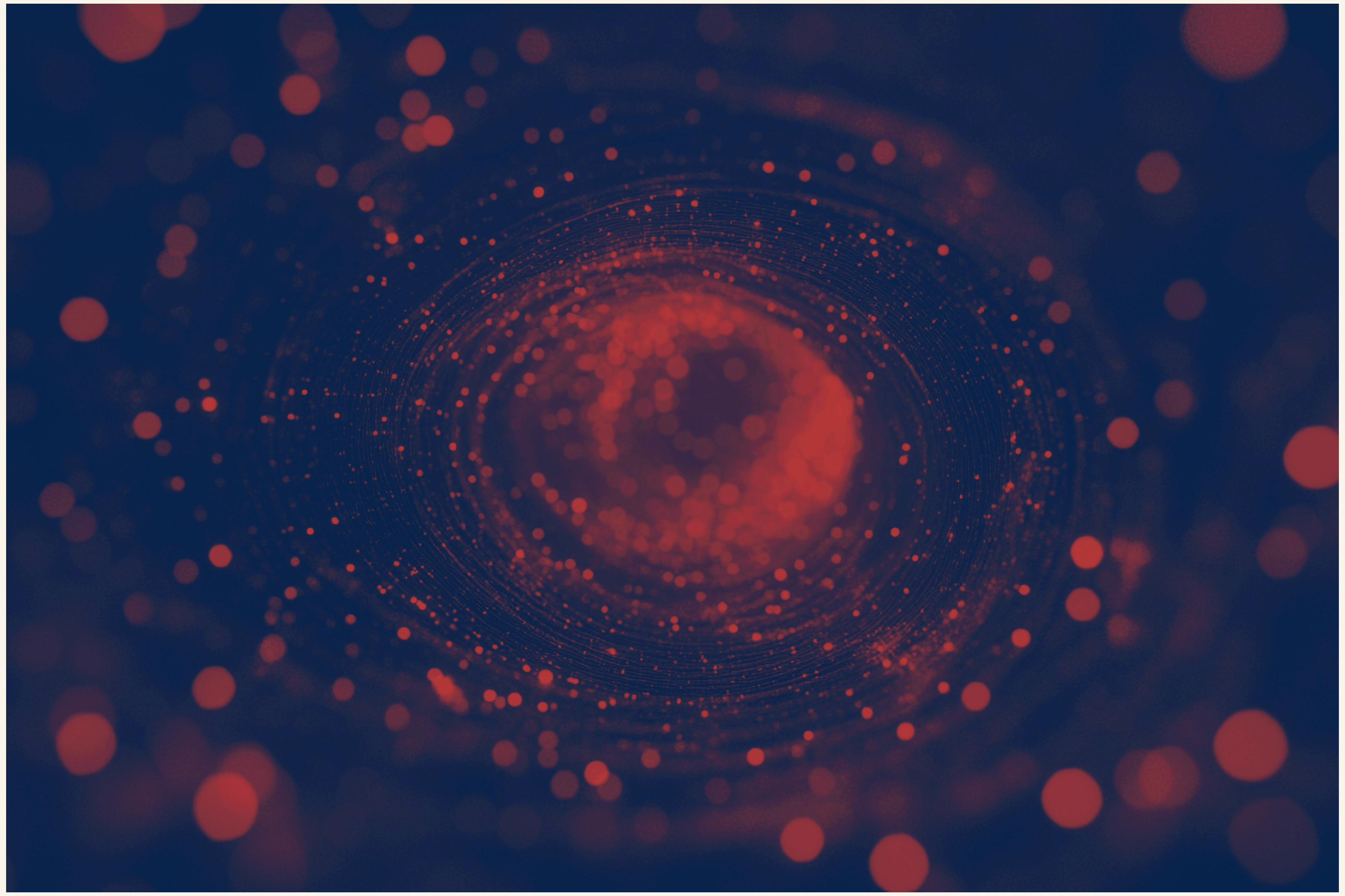
- For initial conditions with $r(0) > 1$, trajectories spiral inward toward the helix as the height z increases.

03

Exponential Convergence

- Convergence to the attractor is **exponential**, with rate $\mu = -2$ confirmed both analytically and numerically.

Main Theorem and Its Meaning



Exponential Convergence to Helix

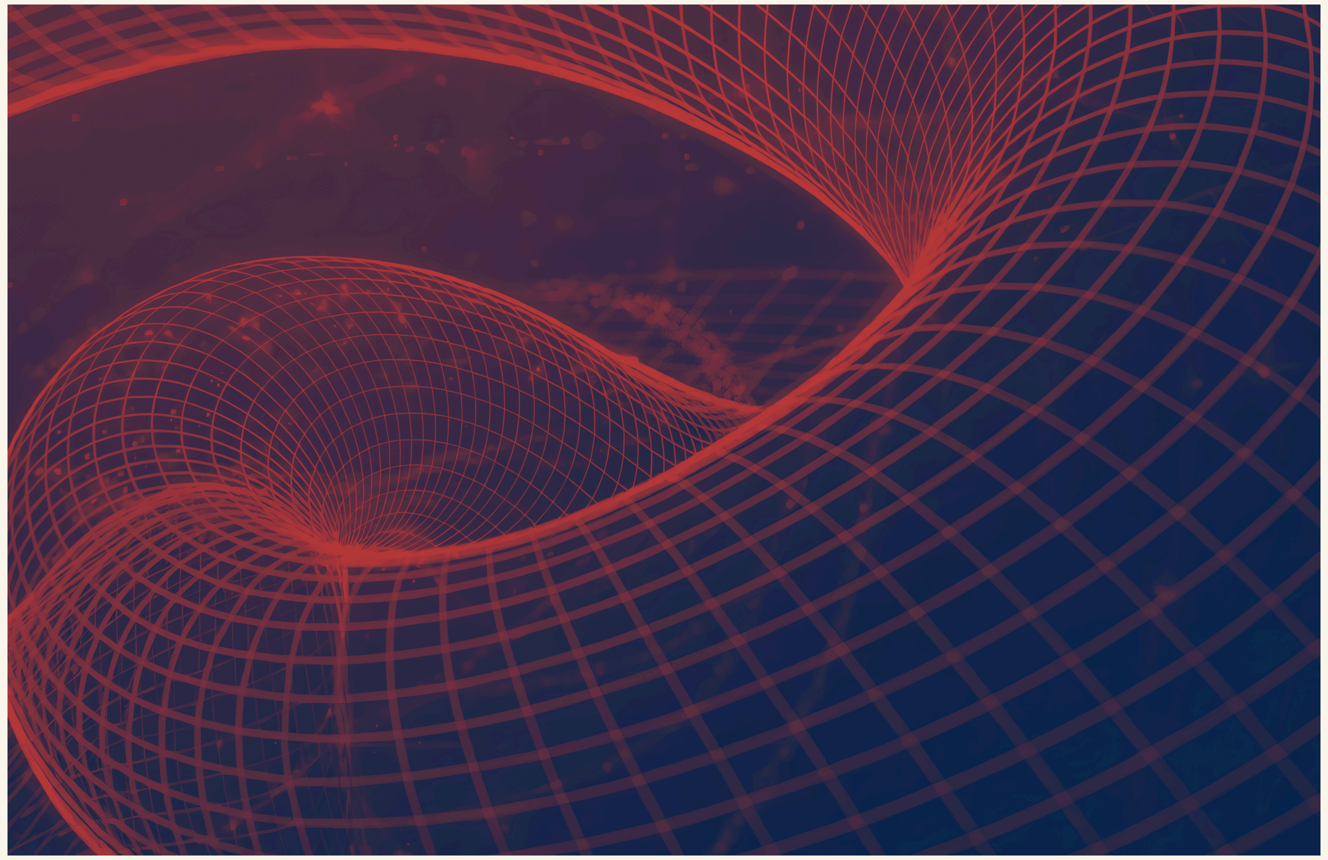
- For any initial radius **$r(0) > 1$**, the system's trajectory rapidly approaches the unit circle $r = 1$.
- Convergence occurs at an exponential rate with **$\mu = -2$**.

Linear Growth and Uniform Rotation

- The height coordinate **$z(t)$** increases linearly over time.
- The angular coordinate **$\theta(t)$** advances at a constant rate, producing a uniform spiral motion.

Spiraling onto the Helical Attractor

- Trajectories spiral upward and inward, settling onto a stable **helical limit set** on the contact manifold.



Numerical Experiments in Action

Precision Numerical Integration

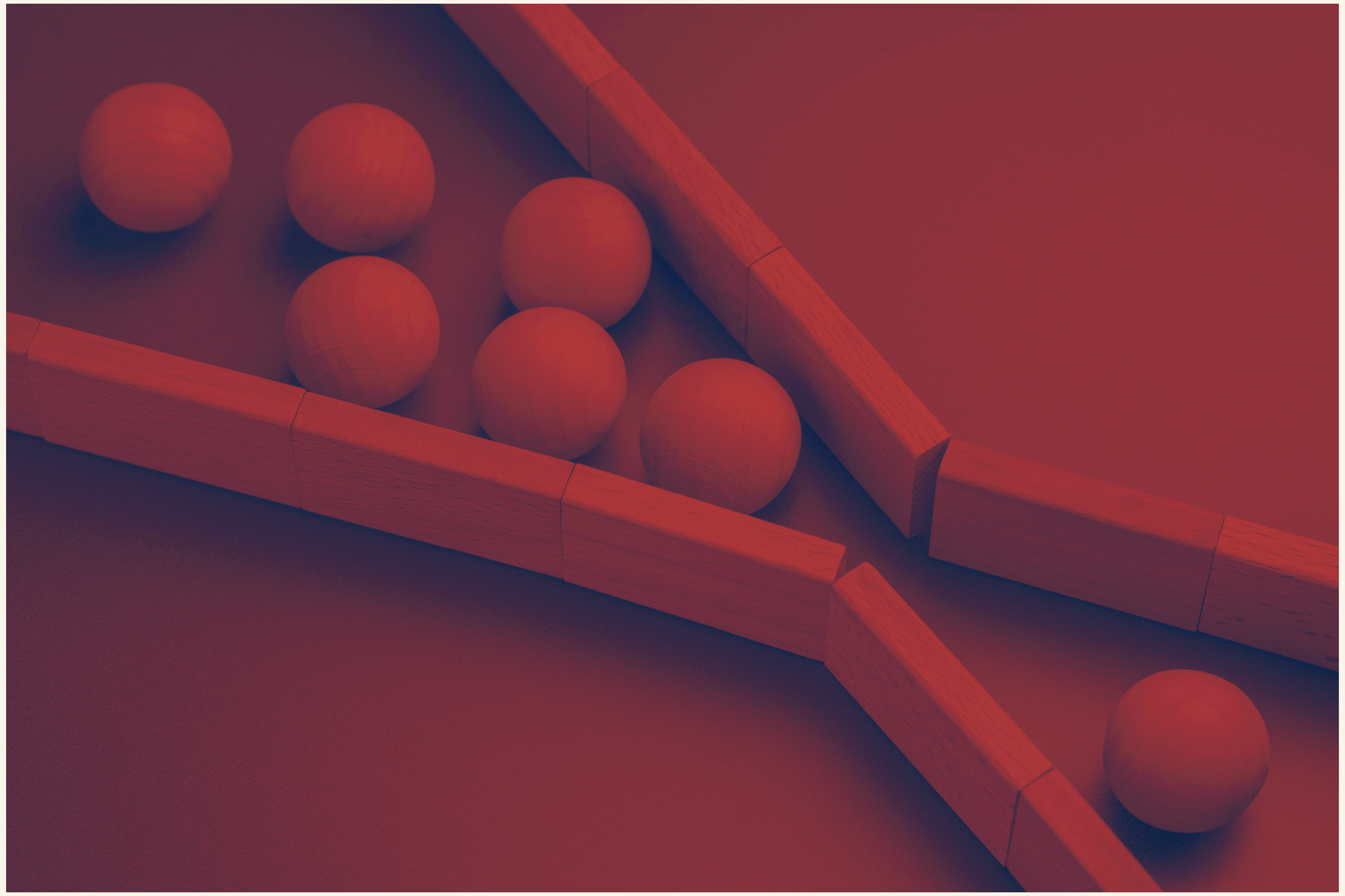
- DOP853 solver with strict tolerances (rtol = 10^{-10}, atol = 10^{-12}) ensures **high accuracy** in simulations.
- Numerical results consistently validate the **theoretical attractor** predictions.

Decay Rate Analysis

- Empirical decay rates for $|r(t) - 1|$ approach **$\mu = -2$** as initial perturbation increases.
- Least-squares fits of $\log|r(t) - 1|$ confirm exponential convergence.

Refining Theoretical Estimates

- Numerical experiments reveal an **asymmetric basin boundary** at $r \approx 0.8$, refining the Gronwall estimate.
- Empirical data highlights the importance of **numerical validation** in dynamical systems.



Basin of Attraction: Surprises

Gronwall Estimate: Too Generous

- Classical estimate predicts convergence for $|r - 1| < 1/3$
- Fails to capture true basin boundary

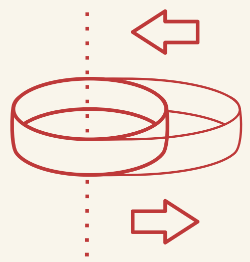
Asymmetric Inner Boundary Revealed

- Numerical results show boundary at **$r^* \approx 0.80$**
- Basin is not symmetric as previously assumed

Collapse for Small Initial Radius

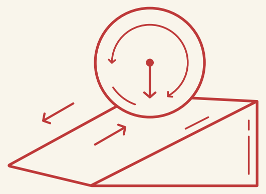
- Trajectories with r(0) □ 0.8 collapse: z □ −∞ in finite time
- Exponential amplification of e^{-z} coupling drives collapse

Geometry Meets Dynamics



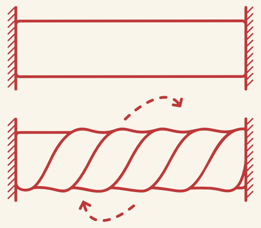
Contact Geometry Drives Dynamics

- Contact manifolds impose a **non-integrable** structure, shaping possible system behaviors.
- Unlike planar systems, trajectories are forced into **helical limit sets** due to this geometry.



Helical Attractors and the Reeb Orbit

- The **Reeb orbit** on the unit cylinder ($r = 1$) acts as a global attractor for the system.
- All trajectories with $r(0) > 1$ spiral towards this helical set, not a simple closed curve.



Non-Integrability and Limit Sets

- Non-integrability prevents the existence of planar periodic orbits, enforcing **helical motion** in 3D.

Formal Verification and Future Directions

Formal Verification with Lean 4

Lean 4 Formalization of Helical Attractors

- Project AXLE encodes the main results using **Lean 4** and **Mathlib 4**

Operator Chain Structures the Proof

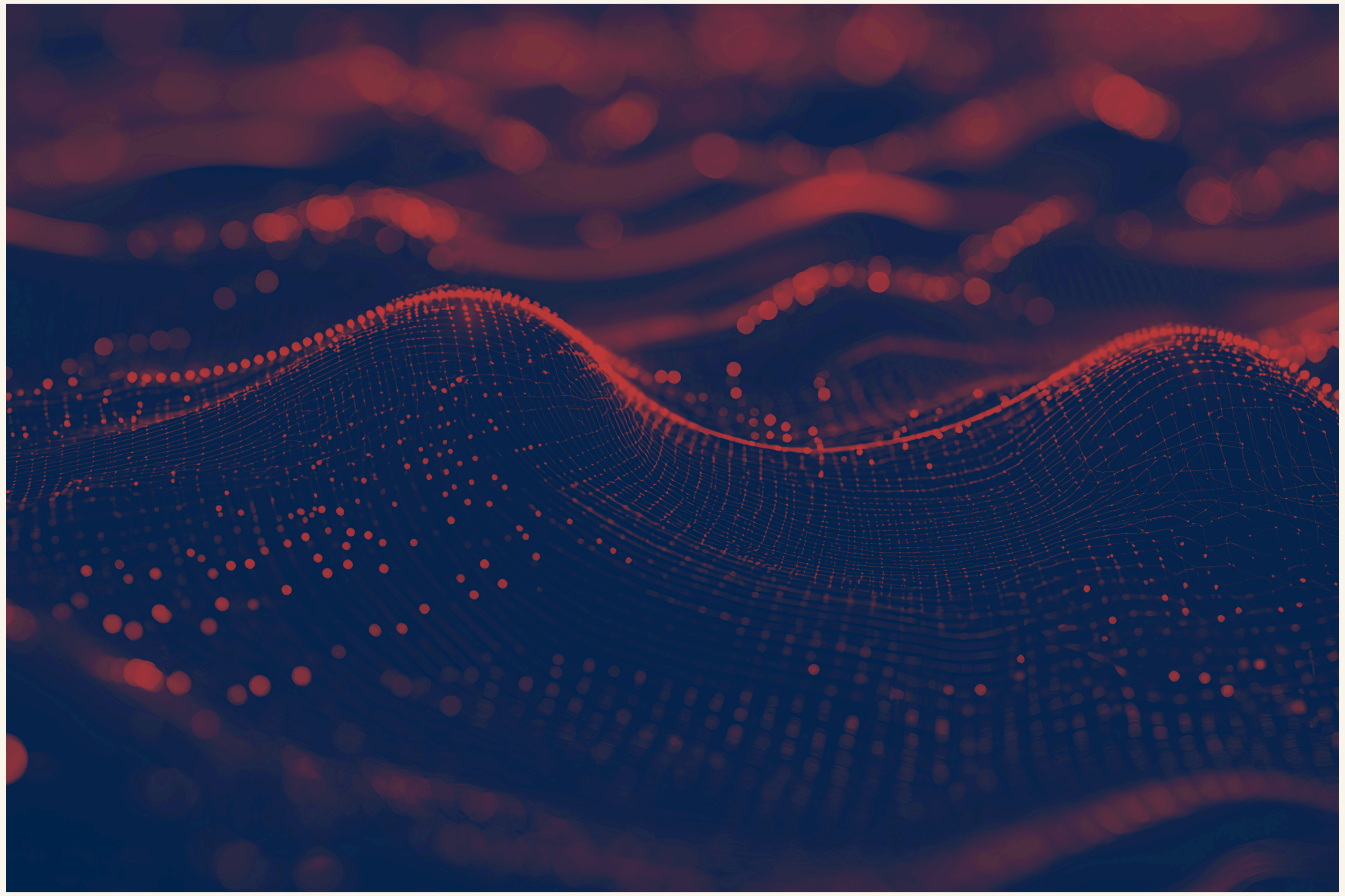
- Proof built via a chain: Compress, Threshold, Fold, Unfold

Open Problems and Research Directions

- Key challenge: formal proof of the **kappa-Lipschitz** property remains open

Accessible and Collaborative

- All code and materials are open access for further exploration and contribution



Open Questions and Research Directions

Analytical Characterization of Basin Boundaries

- Current boundary at $r^* \approx 0.80$ is empirically determined; analytical derivation remains an open challenge.
- Understanding the interplay between nonlinear terms and the destabilizing e^{-z} coupling is key.

Advancing Formal Verification

- Further development of **Lean 4** formalization, especially the unresolved kappa-Lipschitz property.
- Automating proofs for stability and convergence in contact dynamical systems.

Extending to Broader Classes of Systems

- Exploring helical attractors in other contact manifolds and higher-dimensional settings.
- Investigating the impact of different coupling terms and geometries.

Key Takeaways

Helical Attractors in Contact Geometry

- Contact manifolds naturally support **helical limit sets** as attractors.
- The Reeb flow on the unit cylinder ($r = 1$) forms a **stable helix**.

Numerical and Formal Approaches

- High-precision numerics confirm **exponential convergence** to the helix.
- Formal verification in Lean 4 connects theory with computation.

The dm3 System: A Model Example

- Combines Hopf normal form with an **exponentially decaying coupling**.
- Reveals subtle features like **asymmetric basins of attraction**.

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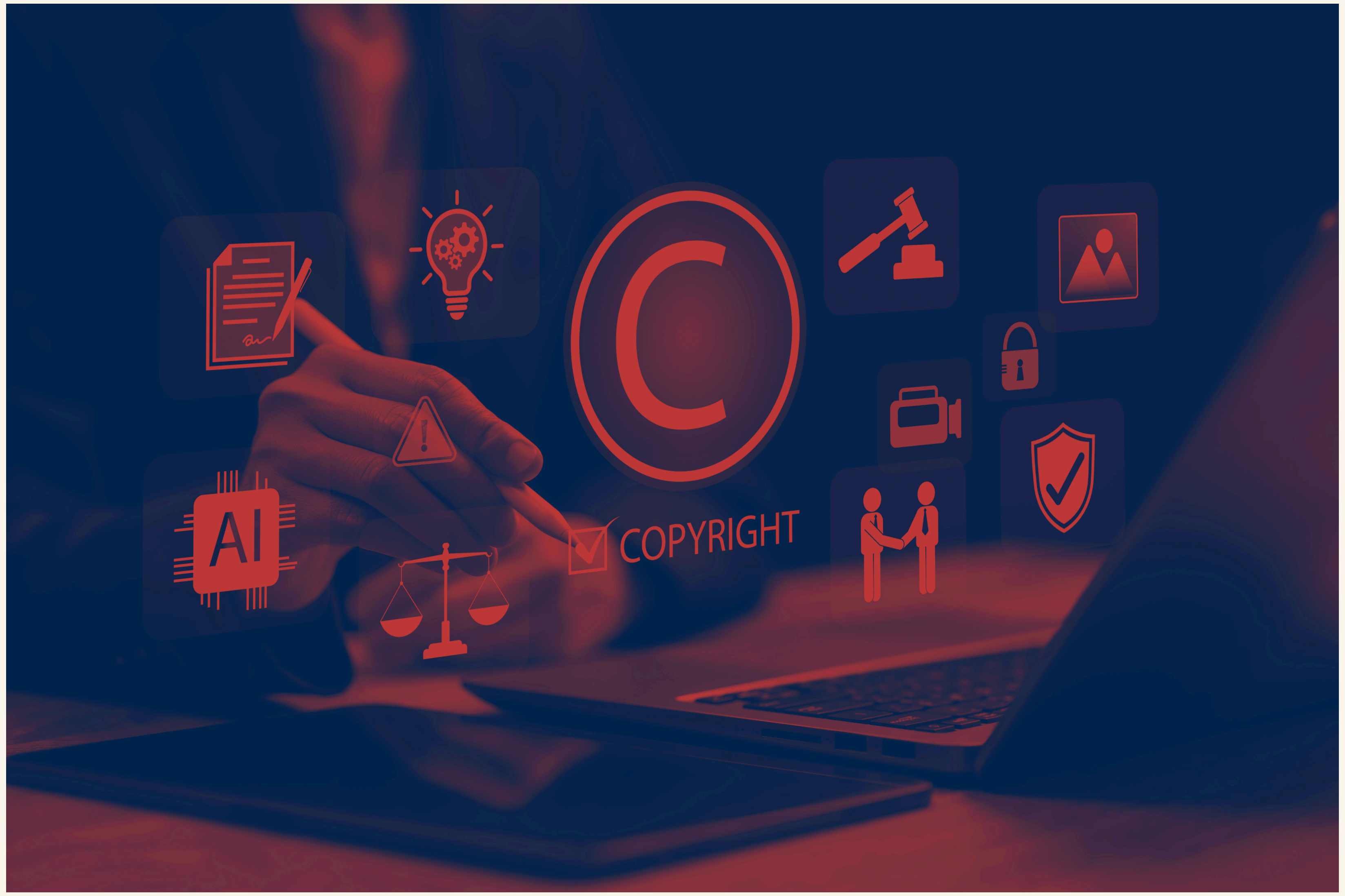

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